

CHARACTERIZATION OF TEST FUNCTIONS IN CKS-SPACE

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ABSTRACT. We prove a characterization theorem for the test functions in a CKS-space. Some crucial ideas concerning the growth condition are given.

1. INTRODUCTION

Let \mathcal{E} be a real countably-Hilbert space with topology given by a sequence of norms $\{|\cdot|_p\}_{p=0}^\infty$ (see [11].) Let \mathcal{E}_p be the completion of \mathcal{E} with respect to the norm $|\cdot|_p$. Assume the following conditions:

- (a) There exists a constant $0 < \rho < 1$ such that $|\cdot|_0 \leq \rho|\cdot|_1 \leq \cdots \leq \rho^p|\cdot|_p \leq \cdots$.
- (b) For any $p \geq 0$, there exists $q > p$ such that the inclusion map $i_{q,p} : \mathcal{E}_q \rightarrow \mathcal{E}_p$ is a Hilbert-Schmidt operator and $\|i_{q,p}\|_{HS} < 1$.

By using the Riesz representation theorem to identify \mathcal{E}_0 with its dual space we get the continuous inclusion maps:

$$\mathcal{E} \subset \mathcal{E}_p \subset \mathcal{E}_0 \subset \mathcal{E}'_p \subset \mathcal{E}', \quad p \geq 0,$$

where \mathcal{E}'_p and \mathcal{E}' are the dual spaces of \mathcal{E}_p and \mathcal{E} , respectively. Condition (b) says that \mathcal{E} is a nuclear space and so $\mathcal{E} \subset \mathcal{E}_0 \subset \mathcal{E}'$ is a Gel'fand triple. Let μ be the standard Gaussian measure on \mathcal{E}' . Let (L^2) denote the Hilbert space of μ -square integrable functions on \mathcal{E}' . By the Wiener-Itô theorem each φ in (L^2) can be uniquely expressed as

$$\varphi = \sum_{n=0}^{\infty} I_n(f_n) = \sum_{n=0}^{\infty} \langle \cdot^{\otimes n}, f_n \rangle, \quad f_n \in \widehat{\mathcal{E}_0^{\otimes n}}, \quad (1.1)$$

and the (L^2) -norm $\|\varphi\|_0$ of φ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n! |f_n|_0^2 \right)^{1/2}.$$

Now, we describe the spaces of test and generalized functions on the space \mathcal{E}' introduced by Cochran et al. in a recent paper [4]. Let $\{\alpha(n)\}_{n=0}^\infty$ be a sequence of numbers satisfying the following conditions:

(A1) $\alpha(0) = 1$ and $\inf_{n \geq 0} \alpha(n) > 0$.

(A2) $\lim_{n \rightarrow \infty} \left(\frac{\alpha(n)}{n!} \right)^{1/n} = 0$.

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Let $\varphi \in (L^2)$ be represented as in Equation (1.1). For each nonnegative integer p , define

$$\|\varphi\|_{p,\alpha} = \left(\sum_{n=0}^{\infty} n! \alpha(n) |f_n|_p^2 \right)^{1/2}. \quad (1.2)$$

Let $[\mathcal{E}_p]_\alpha = \{\varphi \in (L^2); \|\varphi\|_{p,\alpha} < \infty\}$. Define the space $[\mathcal{E}]_\alpha$ of test functions to be the projective limit of the family $\{[\mathcal{E}_p]_\alpha; p \geq 0\}$. Its dual space $[\mathcal{E}]_\alpha^*$ is the space of generalized functions. By identifying (L^2) with its dual we get the following continuous inclusion maps:

$$[\mathcal{E}]_\alpha \subset [\mathcal{E}_p]_\alpha \subset (L^2) \subset [\mathcal{E}_p]_\alpha^* \subset [\mathcal{E}]_\alpha^*, \quad p \geq 0.$$

If $\Phi \in [\mathcal{E}_p]_\alpha^*$ (the dual space of $[\mathcal{E}_p]_\alpha$) is represented by $\Phi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , F_n \rangle$, then its $[\mathcal{E}_p]_\alpha^*$ -norm is given by

$$\|\Phi\|_{-p,1/\alpha} = \left(\sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} |F_n|_{-p}^2 \right)^{1/2}. \quad (1.3)$$

This Gel'fand triple $[\mathcal{E}]_\alpha \subset (L^2) \subset [\mathcal{E}]_\alpha^*$ is called the CKS-space associated with a sequence $\{\alpha(n)\}_{n=0}^{\infty}$ of numbers satisfying the above conditions (A1) and (A2).

Several characterization theorems for generalized functions in $[\mathcal{E}]_\alpha^*$ have been proved in the paper [4]. However, no characterization theorem for test functions in $[\mathcal{E}]_\alpha$ is given. The purpose of the present paper is to prove such a theorem. In addition we will mention some crucial ideas in order to get a complete description of the characterization theorems for test and generalized functions in our ongoing research collaboration project. We remark that similar results have been obtained by Gannoun et al. [5].

2. CHARACTERIZATION THEOREMS

For $\xi \in \mathcal{E}_c$ (the complexification of \mathcal{E}), the renormalized exponential function $:e^{\langle \cdot, \xi \rangle}:$ is defined by

$$:e^{\langle \cdot, \xi \rangle}: = \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} : , \xi^{\otimes n} \rangle. \quad (2.1)$$

For any $p \geq 0$, we have

$$\|:e^{\langle \cdot, x \rangle}:\|_{p,\alpha} = G_\alpha(|\xi|_p^2)^{1/2}, \quad (2.2)$$

where G_α is the exponential generating function of the sequence $\{\alpha(n)\}_{n=0}^{\infty}$, i.e.,

$$G_\alpha(z) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} z^n, \quad z \in \mathbb{C}.$$

By condition (A2) of the sequence $\{\alpha(n)\}_{n=0}^{\infty}$, the function G_α is an entire function. Equation (2.2) implies that $:e^{\langle \cdot, x \rangle}:$ is a test function in $[\mathcal{E}]_\alpha$ for any $\xi \in \mathcal{E}_c$.

The S -transform of a generalized function Φ in $[\mathcal{E}]_\alpha^*$ is the function $S\Phi$ defined on \mathcal{E}_c by

$$S\Phi(\xi) = \langle \langle \Phi, :e^{\langle \cdot, x \rangle} : \rangle \rangle, \quad \xi \in \mathcal{E}_c,$$

where $\langle \langle \cdot, \cdot \rangle \rangle$ is the bilinear pairing of $[\mathcal{E}]_\alpha^*$ and $[\mathcal{E}]_\alpha$.

We state three conditions on the sequence $\{\alpha(n)\}_{n=0}^{\infty}$ as follows:

- (B1) $\limsup_{n \rightarrow \infty} \left(\frac{n!}{\alpha(n)} \inf_{r>0} \frac{G_\alpha(r)}{r^n} \right)^{1/n} < \infty$.
- (B2) The sequence $\gamma(n) = \frac{\alpha(n)}{n!}, n \geq 0$, is log-concave, i.e.,
- $$\gamma(n)\gamma(n+2) \leq \gamma(n+1)^2, \quad \forall n \geq 0.$$
- (B3) The sequence $\{\alpha(n)\}_{n=0}^\infty$ is log-convex, i.e.,
- $$\alpha(n)\alpha(n+2) \geq \alpha(n+1)^2, \quad \forall n \geq 0.$$

Condition (B1) is used in the characterization theorem for generalized functions. By Corollary 4.4 in [4] condition (B2) implies condition (B1). Condition (B3) implies condition (B2) to be defined below.

The following characterization theorem for generalized functions in $[\mathcal{E}]_\alpha^*$ has been proved in [4].

Theorem 2.1. *Suppose the sequence $\{\alpha(n)\}_{n=0}^\infty$ satisfies conditions (A1) and (A2). (I) Let $\Phi \in [\mathcal{E}]_\alpha^*$. Then the S -transform $F = S\Phi$ of Φ satisfies the conditions:*

- (1) *For any ξ, η in \mathcal{E}_c , the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
- (2) *There exist constants $K, a, p \geq 0$ such that*

$$|F(\xi)| \leq KG_\alpha(a|\xi|_p^2)^{1/2}, \quad \xi \in \mathcal{E}_c. \quad (2.3)$$

(II) *Conversely, assume that condition (B1) holds and let F be a function on \mathcal{E}_c satisfying the above conditions (1) and (2). Then there exists a unique $\Phi \in [\mathcal{E}]_\alpha^*$ such that $F = S\Phi$.*

We mention that condition (2) is actually equivalent to the condition: there exist constants $K, p \geq 0$ such that

$$|F(\xi)| \leq KG_\alpha(|\xi|_p^2)^{1/2}, \quad \xi \in \mathcal{E}_c.$$

This fact can be easily checked by using the fact that $|\xi|_p \leq \rho^{q-p}|\xi|_q$ for any $q \geq p$. Having the constant a in condition (2) is for convenience to check whether a given function F satisfies the condition.

By this theorem, if we assume condition (B1), then conditions (1) and (2) are necessary and sufficient for a function F defined on \mathcal{E}_c to be the S -transform of a generalized function in $[\mathcal{E}]_\alpha^*$. As mentioned above, condition (B2) implies condition (B1). Thus under condition (B2), conditions (1) and (2) are also necessary and sufficient for F to be the S -transform of a generalized function in $[\mathcal{E}]_\alpha^*$.

For the characterization theorem for test functions in $[\mathcal{E}]_\alpha$, we need to define the exponential generating function of the sequence $\{\frac{1}{\alpha(n)}\}_{n=0}^\infty$:

$$G_{1/\alpha}(z) = \sum_{n=0}^{\infty} \frac{1}{n!\alpha(n)} z^n, \quad z \in \mathbb{C}, \quad (2.4)$$

Moreover, we need the corresponding conditions (A2) (B1) (B2) for the sequence $\{\frac{1}{\alpha(n)}\}_{n=0}^\infty$:

- (A2) $\lim_{n \rightarrow \infty} \left(\frac{1}{n!\alpha(n)} \right)^{1/n} = 0$.
- (B1) $\limsup_{n \rightarrow \infty} \left(n!\alpha(n) \inf_{r>0} \frac{G_{1/\alpha}(r)}{r^n} \right)^{1/n} < \infty$.
- (B2) The sequence $\{\frac{1}{n!\alpha(n)}\}_{n=0}^\infty$ is log-concave.

It follows from condition $(\tilde{A}2)$ that the exponential generating function $G_{1/\alpha}$ is an entire function. Note that by condition (A1), $\alpha(n) \geq \alpha_0$ for all n , where $\alpha_0 = \inf_{n \geq 0} \alpha(n)$. Hence by the Stirling formula

$$\left(\frac{1}{n!\alpha(n)}\right)^{1/n} \leq \left(\frac{1}{n!\alpha_0}\right)^{1/n} \sim \left(\frac{1}{\alpha_0 \sqrt{2\pi n}}\right)^{1/n} \frac{e}{n} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This shows that condition (A1) implies condition $(\tilde{A}2)$. On the other hand, by applying Corollary 4.4 in [4] to the sequence $\{\frac{1}{\alpha(n)}\}_{n=0}^\infty$ we see that condition $(\tilde{B}2)$ implies condition $(\tilde{B}1)$. Moreover, it is easy to check that condition (B3) implies condition $(\tilde{B}2)$.

Now, we study the characterization theorem for test functions in $[\mathcal{E}]_\alpha$. First we prove a lemma.

Lemma 2.2. *Assume that condition $(\tilde{B}1)$ holds and let F be a function on \mathcal{E}_c satisfying the conditions:*

- (1) *For any ξ, η in \mathcal{E}_c , the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
- (2) *There exist constants $K, a, p \geq 0$ such that*

$$|F(\xi)| \leq KG_{1/\alpha}(a|\xi|_{-p}^2)^{1/2}, \quad \xi \in \mathcal{E}_c.$$

Then there exists a unique $\varphi \in [\mathcal{E}]_\alpha^$ such that $F = S\varphi$. In fact, $\varphi \in [\mathcal{E}_q]_\alpha$ for any $q \in [0, p]$ satisfying the condition*

$$ae^2 \|i_{p,q}\|_{HS}^2 \limsup_{n \rightarrow \infty} \left(n! \alpha(n) \inf_{r>0} \frac{G_{1/\alpha}(r)}{r^n} \right)^{1/n} < 1 \quad (2.5)$$

and

$$\|\varphi\|_{q,\alpha}^2 \leq \frac{K^2}{2\pi} \sum_{n=0}^\infty \left(n! \alpha(n) \inf_{r>0} \frac{G_{1/\alpha}(r)}{r^n} \right) \left(ae^2 \|i_{p,q}\|_{HS}^2 \right)^n. \quad (2.6)$$

Proof. We can modify the proof of Theorem 8.9 in [11]. Use the analyticity in condition (1) to get the expansion

$$F(\xi) = \sum_{n=0}^\infty \langle f_n, \xi^{\otimes n} \rangle.$$

Then use the Cauchy formula and condition (2) to show that for any ξ_1, \dots, ξ_n in $[\mathcal{E}_p]_\alpha^*$ and any $R > 0$:

$$|\langle f_n, \xi_1 \hat{\otimes} \dots \hat{\otimes} \xi_n \rangle| \leq \frac{1}{n!} \frac{KG_{1/\alpha}(an^2 R^2)^{1/2}}{R^n} |\xi_1|_{-p} \dots |\xi_n|_{-p}.$$

Make a change of variables $r = an^2 R^2$ and use the inequality $n^n \leq n!e^n/\sqrt{2\pi}$ (see page 357 in [11]) and then take the infimum over $r > 0$ to get

$$|\langle f_n, \xi_1 \hat{\otimes} \dots \hat{\otimes} \xi_n \rangle|^2 \leq \frac{K^2}{2\pi} a^n e^{2n} \left(\inf_{r>0} \frac{G_{1/\alpha}(r)}{r^n} \right) |\xi_1|_{-p}^2 \dots |\xi_n|_{-p}^2.$$

Now, suppose $q \in [0, p]$ satisfies the condition in Equation (2.5). Then by similar arguments as in the proofs of Theorems 8.2 and 8.9 in [11] we can derive

$$|f_n|_q^2 \leq \frac{K^2}{2\pi} a^n e^{2n} \left(\inf_{r>0} \frac{G_{1/\alpha}(r)}{r^n} \right) \|i_{p,q}\|_{HS}^{2n}.$$

Therefore

$$\begin{aligned}\|\varphi\|_{q,\alpha}^2 &= \sum_{n=0}^{\infty} n! \alpha(n) |f_n|_q^2 \\ &\leq \frac{K^2}{2\pi} \sum_{n=0}^{\infty} \left(n! \alpha(n) \inf_{r>0} \frac{G_{1/\alpha}(r)}{r^n} \right) \left(ae^2 \|i_{p,q}\|_{HS}^2 \right)^n.\end{aligned}$$

Note that the series converges because of the condition in Equation (2.5). \square

Theorem 2.3. *Suppose the sequence $\{\alpha(n)\}_{n=0}^{\infty}$ satisfies conditions (A1).*

(I) *Let $\varphi \in [\mathcal{E}]_{\alpha}$. Then the S -transform $F = S\varphi$ of φ satisfies the conditions:*

- (1) *For any ξ, η in \mathcal{E}_c , the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
- (2) *For any constants $a, p \geq 0$ there exists a constant $K \geq 0$ such that*

$$|F(\xi)| \leq KG_{1/\alpha}(a|\xi|_{-p}^2)^{1/2}, \quad \xi \in \mathcal{E}_c. \quad (2.7)$$

(II) *Conversely, assume that condition $(\tilde{B}1)$ holds and let F be a function on \mathcal{E}_c satisfying the above conditions (1) and (2). Then there exists a unique $\varphi \in [\mathcal{E}]_{\alpha}$ such that $F = S\varphi$.*

We remark that condition (2) is actually equivalent to the condition: for any constant $p \geq 0$ there exists a constant $K \geq 0$ such that

$$|F(\xi)| \leq KG_{1/\alpha}(|\xi|_{-p}^2)^{1/2}, \quad \xi \in \mathcal{E}_c.$$

This fact can be easily checked by using the fact that $|\xi|_{-q} \leq \rho^{q-p} |\xi|_{-p}$ for any $q \geq p$. Having the constant a in condition (2) is for convenience to check whether a given function F satisfies the condition.

Proof. Let $\varphi \in [\mathcal{E}]_{\alpha}$. Note that $[\mathcal{E}]_{\alpha} \subset [\mathcal{E}]_{\alpha}^*$ and so $\varphi \in [\mathcal{E}]_{\alpha}^*$. Thus $F = S\varphi$ satisfies condition (1) by Theorem 2.1. To check condition (2), note that

$$F(\xi) = S\varphi(\xi) = \langle \langle :e^{\langle \cdot, \xi \rangle} : , \varphi \rangle \rangle, \quad \xi \in \mathcal{E}_c.$$

Since $\varphi \in [\mathcal{E}]_{\alpha}$, we have $\|\varphi\|_{q,\alpha} < \infty$ for all $q \geq 0$. Hence

$$|F(\xi)| \leq \|\varphi\|_{q,\alpha} \| :e^{\langle \cdot, \xi \rangle} : \|_{-q,1/\alpha}.$$

But by Equations (1.3) and (2.1) $\| :e^{\langle \cdot, \xi \rangle} : \|_{-q,1/\alpha} = G_{1/\alpha}(|\xi|_{-q}^2)^{1/2}$. Hence

$$|F(\xi)| \leq \|\varphi\|_{q,\alpha} G_{1/\alpha}(|\xi|_{-q}^2)^{1/2}.$$

For any given $a, p \geq 0$, choose $q \geq p$ such that $\rho^{q-p} \leq \sqrt{a}$. Then

$$|\xi|_{-q} \leq \rho^{q-p} |\xi|_{-p} \leq \sqrt{a} |\xi|_{-p}.$$

Therefore, we obtain

$$|F(\xi)| \leq \|\varphi\|_{q,\alpha} G_{1/\alpha}(|\xi|_{-q}^2)^{1/2} \leq \|\varphi\|_{q,\alpha} G_{1/\alpha}(a|\xi|_{-p}^2)^{1/2}.$$

To prove the converse, assume that condition $(\tilde{B}1)$ holds and let F be a function on \mathcal{E}_c satisfying conditions (1) and (2). Let $q \geq 0$ be any given number. Choose $a, p \geq 0$ such that

$$ae^2 \|i_{p,q}\|_{HS}^2 \limsup_{n \rightarrow \infty} \left(n! \alpha(n) \inf_{r>0} \frac{G_{1/\alpha}(r)}{r^n} \right)^{1/n} < 1.$$

This inequality can be achieved in two ways: (1) choose any $a \geq 0$ and then use the fact that $\lim_{p \rightarrow \infty} \|i_{p,q}\|_{HS} = 0$, (2) choose any $p \geq 0$ such that $i_{p,q}$ is a Hilbert-Schmidt operator and then choose a sufficiently small number $a \geq 0$. (For the first way we can choose $a = 1$ and this is exactly the fact mentioned in the above remark.) With the chosen a and p , use condition (2) to get a constant K such that the inequality in Equation (2.7) holds. Then we apply Lemma 2.2 to get the inequality in Equation (2.6) so that $\|\varphi\|_{q,\alpha} < \infty$. Hence $\varphi \in [\mathcal{E}_q]_\alpha$ for all $q \geq 0$. Thus $\varphi \in [\mathcal{E}]_\alpha$ and the converse of the theorem is proved. \square

3. EXAMPLES AND COMMENTS

1. Four conditions

In the definition of CKS-space and the characterization theorems of generalized and test functions we have assumed several conditions on the sequence $\{\alpha(n)\}_{n=0}^\infty$: (A1), (A2), (B1), (B2), (B3), $(\tilde{A}2)$, $(\tilde{B}1)$, $(\tilde{B}2)$. Recall that we have the following implications:

$$(A1) \implies (\tilde{A}2), \quad (B2) \implies (B1), \quad (\tilde{B}2) \implies (\tilde{B}1), \quad (B3) \implies (\tilde{B}2).$$

Taking these implications into account we will consider below the **four conditions**: (A1), (A2), (B2), (B3).

2. Examples

We give three examples corresponding to the Hida-Kubo-Takenaka, Kondratiev-Streit, and CKS-spaces.

Example 3.1. For the Hida-Kubo-Takenaka space $(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*$ (see [6] [9] [10] [13],) the sequence is given by $\alpha(n) \equiv 1$. Obviously, this sequence satisfies the above four conditions. The corresponding exponential generating functions are

$$G_\alpha(r) = e^r, \quad G_{1/\alpha}(r) = e^r.$$

Thus the growth conditions in Equations (2.3) and (2.7) can be stated as

$$|F(\xi)| \leq K e^{a|\xi|_p^2}, \quad |F(\xi)| \leq K e^{a|\xi|_{-p}^2}.$$

Theorems 2.1 and 2.3 are due to Potthoff-Streit [14] and Kuo-Potthoff-Streit [12], respectively.

Example 3.2. For the Kondratiev-Streit space $(\mathcal{E})_\beta \subset (L^2) \subset (\mathcal{E})_\beta^*$ (see [7] [8] [11],) the sequence is given by $\alpha(n) = (n!)^\beta$. It is easy to check that this sequence satisfies the above four conditions. The corresponding exponential generating functions are

$$G_\alpha(r) = \sum_{n=0}^{\infty} \frac{1}{(n!)^{1-\beta}} r^n, \quad G_{1/\alpha}(r) = \sum_{n=0}^{\infty} \frac{1}{(n!)^{1+\beta}} r^n.$$

But from page 358 in [11] and Lemma 7.1 (page 61) in [4] we have the inequalities:

$$\exp \left[(1 - \beta) r^{\frac{1}{1-\beta}} \right] \leq G_\alpha(r) \leq 2^\beta \exp \left[(1 - \beta) 2^{\frac{\beta}{1-\beta}} r^{\frac{1}{1-\beta}} \right]. \quad (3.1)$$

On the other hand, from page 358 in [11] and the same argument as in the proof of Lemma 7.1 in [4] we can derive the inequalities:

$$2^{-\beta} \exp \left[(1 + \beta) 2^{-\frac{\beta}{1+\beta}} r^{\frac{1}{1+\beta}} \right] \leq G_{1/\alpha}(r) \leq \exp \left[(1 + \beta) r^{\frac{1}{1+\beta}} \right]. \quad (3.2)$$

The inequalities in Equations (3.1) and (3.2) imply that the growth conditions in Theorems 2.1 and 2.3 are respectively equivalent to the conditions:

- There exist constants $K, a, p \geq 0$ such that

$$|F(\xi)| \leq K \exp \left[a |\xi|_p^{\frac{2}{1+\beta}} \right], \quad \xi \in \mathcal{E}_c.$$

- For any constants $a, p \geq 0$ there exists a constant $K \geq 0$ such that

$$|F(\xi)| \leq K \exp \left[a |\xi|_p^{\frac{2}{1+\beta}} \right], \quad \xi \in \mathcal{E}_c.$$

These two inequalities are the growth conditions used by Kondratiev and Streit in [7] [8] (see also Theorems 8.2 and 8.10 in [11].)

Example 3.3. (Bell's numbers) For each integer $k \geq 2$, consider the k -th iterated exponential function $\exp_k(z) = \exp(\exp(\cdots(\exp(z))))$. This function is entire and so it has the power series expansion

$$\exp_k(z) = \sum_{n=0}^{\infty} \frac{B_k(n)}{n!} z^n.$$

The k -th order Bell's numbers $\{b_k(n)\}_{n=0}^{\infty}$ are defined by

$$b_k(n) = \frac{B_k(n)}{\exp_k(0)}, \quad n \geq 0.$$

Obviously, this sequence $\{b_k(n)\}_{n=0}^{\infty}$ satisfies conditions (A1) and (A2). It has been shown recently in [1] that this sequence satisfies conditions (B2) and (B3). The corresponding exponential generating functions are given by

$$G_{b_k}(r) = \sum_{n=0}^{\infty} \frac{b_k(n)}{n!} r^n = \frac{\exp_k(r)}{\exp_k(0)}, \quad (3.3)$$

$$G_{1/b_k}(r) = \sum_{n=0}^{\infty} \frac{1}{n! b_k(n)} r^n. \quad (3.4)$$

The function $\exp_k(r)$ in Equation (3.3) can be used to give the growth condition in Theorem 2.1 for generalized functions, i.e.,

- There exist constants $K, a, p \geq 0$ such that

$$|F(\xi)| \leq K \left(\exp_k(a |\xi|_p^2) \right)^{1/2}, \quad \xi \in \mathcal{E}_c.$$

On the other hand, we cannot express the sum of the series in Equation (3.4) as an elementary function. Thus the growth condition (2) in Theorem 2.3 is very hard, if not impossible, to verify. Hence it is desirable to find similar inequalities as those in Equation (3.2) for the sequence $\{b_k(n)\}_{n=0}^{\infty}$.

3. General growth order

Being motivated by Examples 3.2 and 3.3, we consider the question: What are the possible functions U and u such that the growth conditions in Theorems 2.1 and 2.3 can be respectively stated as follows?

- There exist constants $K, a, p \geq 0$ such that

$$|F(\xi)| \leq KU \left(a |\xi|_p^2 \right)^{1/2}, \quad \xi \in \mathcal{E}_c.$$

- For any constants $a, p \geq 0$ there exists a constant $K \geq 0$ such that

$$|F(\xi)| \leq Ku(a|\xi|_{-p}^2)^{1/2}, \quad \xi \in \mathcal{E}_c.$$

The answer to this question will be given in our forthcoming papers [2] [3]. In particular, when $\alpha(n) = b_k(n)$, condition (2) in Theorem 2.3 can be replaced by the following growth condition:

- For any constants $a, p \geq 0$ there exists a constant $K \geq 0$ such that

$$|F(\xi)| \leq K \exp \left[\sqrt{a|\xi|_{-p}^2 \log_{k-1} (a|\xi|_{-p}^2)} \right], \quad \xi \in \mathcal{E}_c,$$

where \log_j , $j \geq 1$, is the function defined by

$$\log_1(x) = \log(\max\{x, e\}), \quad \log_j(x) = \log_1(\log_{j-1}(x)), \quad j \geq 2.$$

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